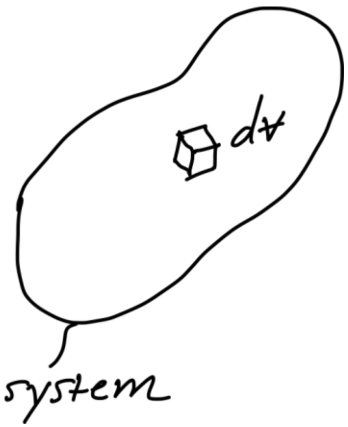
The energy equation



1st law of thermodynamics

$$\underbrace{\frac{D}{Dt}\int e\rho\, d\forall}_{\text{time rate of change of the total energy stored in the system}} = \underbrace{\left(\Sigma \dot{Q}_{in} - \Sigma \dot{Q}_{out}\right)_{sys}}_{\text{net time rate of energy added by heat transfer into the system}} + \underbrace{\left(\Sigma \dot{W}_{in} - \Sigma \dot{W}_{out}\right)_{sys}}_{\text{net time rate of energy added by work transfer into the system}}$$

where

$$e = \check{u} + \frac{V^2}{2} + gz$$

total energy per unit mass

internal energy per unit mass ($\check{u} = \check{c}T$)

specific heat (incompressible)

kinetic energy per unit mass

potential energy per unit mass

R.T.T.: $\dfrac{\partial}{\partial t}\displaystyle\int_{CV} e\rho\, d\forall + \int_{CS} e\rho\, \vec{V}\cdot\vec{n}\, dA = \dot{Q}_{net,in} + \dot{W}_{net,in}$

For steady flows:

$$\int_{CS} \left(\check{u} + \frac{V^2}{2} + gz\right)\rho\, \vec{V}\cdot\vec{n}\, dA = \dot{Q}_{net,in} + \dot{W}_{net,in} \quad (1)$$

Special case: adiabatic flows

$\rightarrow$ no heat transfer $\rightarrow$ $\dot{Q}_{net,in} = 0$

$\dot{W}$ = work per unit time = Power

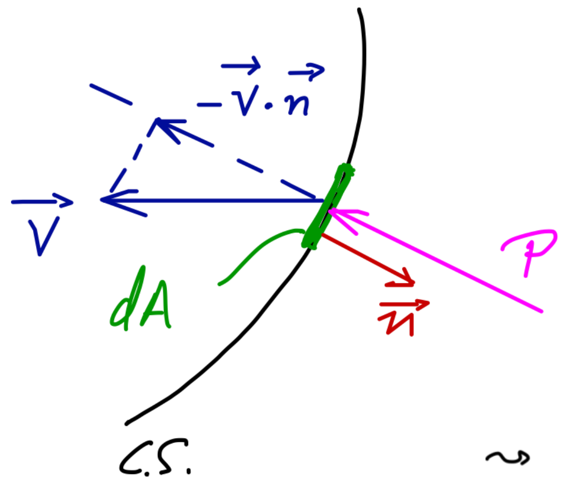
Usually $\dot{W}$ is produced or absorbed by a machine with a rotating shaft (ie., pump, turbine, etc.)

$$\dot{W} = T \cdot \omega$$

rotational speed

torque

Work is also produced by pressure:



The element dA will move inwards with velocity $-\vec{V}\cdot\vec{n}$

So, the work done by pressure on dA:

$$d\dot{W}_p = (p\cdot dA)\cdot(-\vec{V}\cdot\vec{n})$$

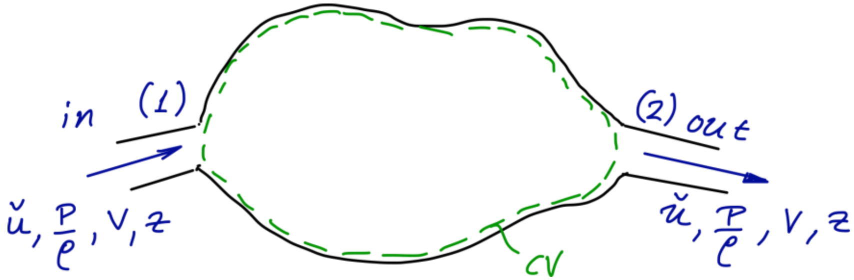
$$\rightsquigarrow \quad \underline{\underline{d\dot{W}_p = -p\cdot\vec{V}\cdot\vec{n}\,dA}}$$

Finally, $\dot{W}_{net,in} = \dot{W}_{shaft} - \int_{cs} p\cdot\vec{V}\cdot\vec{n}\cdot dA \quad (2)$

Substituting into (1):

$$\int_{cs} \rho\left(\check{u} + \frac{V^2}{2} + gz\right)\vec{V}\cdot\vec{n}\,dA = \dot{Q}_{net,in} + \dot{W}_{shaft} - \int_{cs} p\,\vec{V}\cdot\vec{n}\,dA$$

$$\Rightarrow \underline{\underline{\int_{cs} \rho\left(\check{u} + \frac{p}{\rho} + \frac{V^2}{2} + gz\right)\vec{V}\cdot\vec{n}\,dA = \dot{Q}_{net,in} + \dot{W}_{shaft}}}$$



Special case: when there is only one stream entering and leaving the CV, and $\check{u}, \frac{P}{\rho}, \frac{V^2}{2}, z$ are uniform over the inlet and outlet cross-sectional areas, then

$$\int_{CS} \left(\check{u} + \frac{P}{\rho} + \frac{V^2}{2} + gz\right) \rho \vec{V} \cdot \vec{n} \, dA$$

$$= \left(\check{u} + \frac{P}{\rho} + \frac{V^2}{2} + gz\right)_{out} \dot{m}_{out} - \left(\check{u} + \frac{P}{\rho} + \frac{V^2}{2} + gz\right)_{in} \cdot \dot{m}_{in}$$

$$= \dot{m} \left[\check{u}_{out} - \check{u}_{in} + \left(\frac{P}{\rho}\right)_{out} - \left(\frac{P}{\rho}\right)_{in} + \frac{V_{out}^2}{2} - \frac{V_{in}^2}{2} + g\left(z_{out} - z_{in}\right)\right]$$

In such a case, a convenient form of the energy equation is:

$$\dot{m} \left[\check{u}_{out} - \check{u}_{in} + \left(\frac{P}{\rho}\right)_{out} - \left(\frac{P}{\rho}\right)_{in} + \frac{V_{out}^2}{2} - \frac{V_{in}^2}{2} + g\left(z_{out} - z_{in}\right)\right]$$

$$= \dot{Q}_{net,in} + \dot{W}_{shaft}$$

Comparison with the Bernoulli equation

If the flow is incompressible ($\rho = ct$)
and no shaft work is involved ($\dot{W}_{shaft} = 0$)

$$\underbrace{\frac{P_{out}}{\rho} + \frac{V_{out}^2}{2} + g z_{out} = \underbrace{\frac{P_{in}}{\rho} + \frac{V_{in}^2}{2} + g z_{in}}_{\text{available energy}}}_{\text{Bernoulli equation}} - \underbrace{\left(\check{u}_{out} - \check{u}_{in} - q_{net,in}\right)}_{\substack{\text{loss} \geq 0 \\ \text{energy loss} \\ \text{due to friction}}}$$

where $q_{net,in} = \dfrac{\dot{Q}_{net,in}}{\dot{m}}$

In general, for incompressible fluid:

$$\frac{P_{out}}{\rho} + \frac{V_{out}^2}{2} + g z_{out} = \frac{P_{in}}{\rho} + \frac{V_{in}^2}{2} + g z_{in} + w_{shaft} - loss$$

The loss term is often expressed as:

$$loss = K_L \frac{V^2}{2} \qquad (K_L = \text{loss coefficient})$$

and $$w_{shaft} = \frac{\dot{W}_{shaft}}{\dot{m}}$$

$$\frac{P_{out}}{\rho} + \frac{V_{out}^2}{2} + g z_{out} = \frac{P_{in}}{\rho} + \frac{V_{in}^2}{2} + g z_{in} + w_{shaft} - loss$$

Divide by g and let $1 = in$ and $2 = out$:

$$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + z_1 + \underbrace{\frac{w_{shaft}}{g}}_{\substack{h_s \\ \text{"shaft head"}}} = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + z_2 + \underbrace{\frac{loss}{g}}_{\substack{h_L \\ \text{"head loss"}}}$$

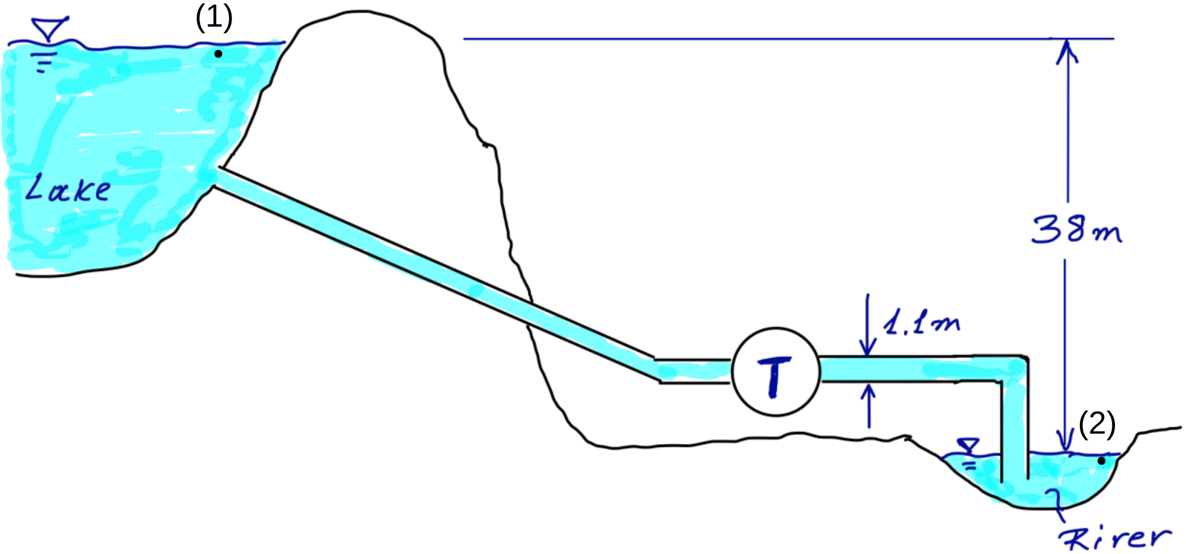
Where,

$$h_s = \text{shaft head} \begin{cases} = h_p & \text{(pump head)} \\ = -h_T & \text{(turbine head)} \end{cases}$$

$$\dot{W}_{shaft} = w_{shaft} \cdot \dot{m} = (h_s \cdot g) \cdot (\rho Q) = \underline{\underline{h_s \cdot \gamma \cdot Q}}$$

$$h_L = \text{head loss} = K_L \frac{V^2}{2g} \quad \text{(always positive)}$$

Example 1



The turbine delivers to the generator 2.8 MW of power when the flowrate is $Q = 15\ m^3/s$.
What is the efficiency, η of the turbine?
Assume a friction loss coefficient $K_L = 1$.

Energy between (1) and (2)

$$\underbrace{\cancel{\frac{P_1}{\gamma}}}_{0} + \underbrace{\cancel{\frac{V_1^2}{2g}}}_{0} + z_1 - h_T = \underbrace{\cancel{\frac{P_2}{\gamma}}}_{0} + \underbrace{\cancel{\frac{V_2^2}{2g}}}_{0} + z_2 + h_L$$

$$\Rightarrow \quad h_T = (z_1 - z_2) - h_L \qquad (1)$$

$$h_L = K_L \frac{V^2}{2g} = 1 \times \frac{\left(\frac{15\ m^3/s}{\pi\ 0.55^2\ m^2} \right)^2}{2 \times 9.81\ m/s^2} = 12.7\ m$$

$(1) \Rightarrow \quad h_T = 38\,m - 12.7\,m = \underline{\underline{25.3\ m}}$

$$\dot{W}_T = \gamma \cdot Q \cdot h_T = 9.8\ kN/m^3 \times 15\ m^3/s \times 25.3\ m$$

$$\Rightarrow \quad \dot{W}_T = 3'720'000\ N \cdot m/s$$

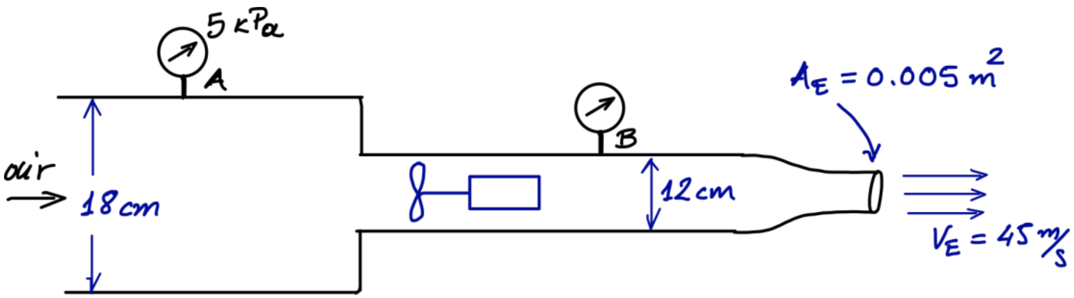
$$\text{or} \quad \dot{W}_T = \underline{\underline{3.72\ MW}}$$

Efficiency $\eta = \dfrac{\text{Power output}}{\text{Power available to turbine}}$

$$\Rightarrow \quad \eta = \frac{2.8\ MW}{3.72\ MW} = 0.753$$

$$\text{or} \quad \underline{\underline{\eta = 75.3\ \%\ \text{efficient}}}$$

Example 2



- Does the propeler act as a pump or as a turbine?
- Determine the power of the device.
- $P_B = ?$

Assume frictionless flow

Continuity: $V_A \cdot A_A = V_B \cdot A_B = V_E \cdot A_E$

$$\Rightarrow V_A = V_E \cdot \frac{A_E}{A_A} = 45\ m/s \cdot \frac{0.005\ m^2}{\pi\ 0.09^2\ m^2} = \underline{8.84\ m/s}$$

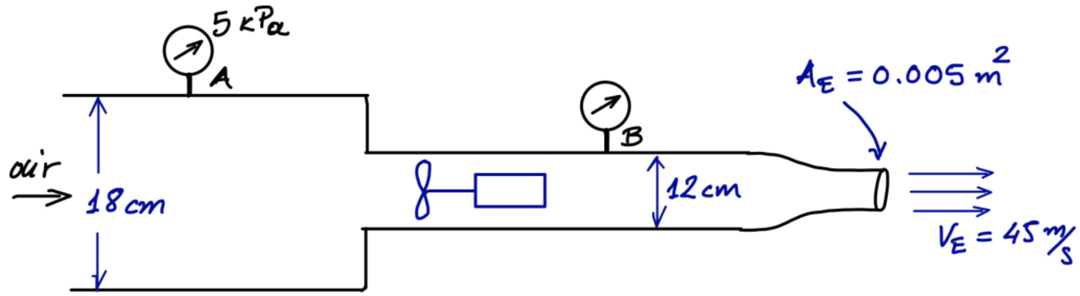
$$\text{and } V_B = V_E\ \frac{A_E}{A_B} = 45\ m/s \cdot \frac{0.005\ m^2}{\pi \cdot 0.06^2\ m^2} = \underline{19.9\ m/s}$$

Energy between A and E (exit):

$$\frac{P_A}{\gamma} + \frac{V_A^2}{2g} + \cancel{z_A} + h_s = \cancel{\frac{P_E}{\gamma}} \to 0 + \frac{V_E}{2g} + \cancel{z_E} + \cancel{h_L} \to 0$$

$$h_s = -\frac{P_A}{\gamma} + \frac{1}{2g}\left(V_E^2 - V_A^2\right) = -\frac{5000\ N/m^2}{12\ N/m^3} + \frac{1}{2 \times 9.81\ m/s^2}\left(45^2 - 8.84^2\right) m^2/s^2$$

$$\Rightarrow h_s = \underline{-317\ m} \qquad \underline{\underline{\text{turbine}}}$$



$$\dot{W}_T = \gamma \cdot Q \cdot h_T = 12\ N/m^3 \cdot (45\ m/s \times 0.005\ m^2) \cdot 317\ m$$

$$= \underline{857\ W}$$

Bernoulli between B and E:

$$P_B + \frac{1}{2}\rho V_B^2 + \cancel{\gamma z_B} = \underbrace{\cancel{P_E}}_{\to 0} + \frac{1}{2}\rho V_E^2 + \cancel{\gamma z_E}$$

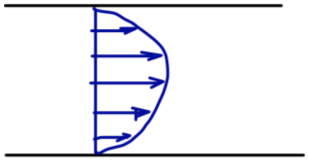
$$\Rightarrow P_B = \frac{1}{2}\rho\left(V_E^2 - V_B^2\right)$$

$$= \frac{1}{2}\, 1.23\ kg/m^3 \cdot \left(45^2 - 19.9^2\right) m^2/s^2$$

$$= 1000\ N/m^2$$

or $\underline{\underline{P_B = 1\ kPa}}$

Non-uniform velocity



In most real cases, the velocity profile at the entry and exit points is <u>not</u> uniform (i.e. parabolic, flattened, etc.)

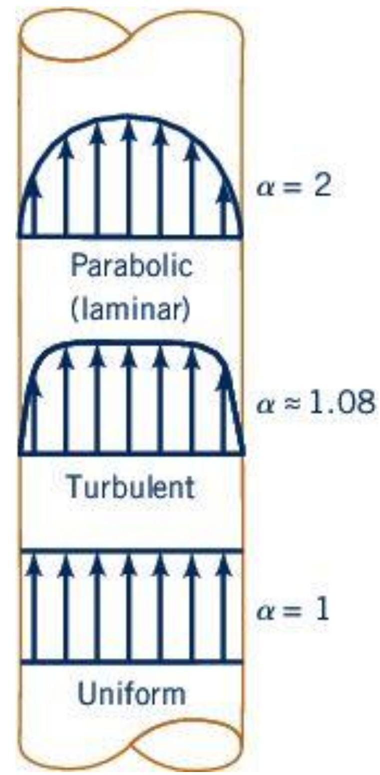
Energy equation:

$$\int_{cs} \rho\left(\check{u} + \frac{P}{\rho} + \frac{V^2}{2} + gz\right)\vec{V}\cdot\vec{n}\,dA = \dot{Q}_{net,in} + \dot{W}_{shaft}$$

The term $\displaystyle\int_{cs} \rho\frac{V^2}{2}\vec{V}\cdot\vec{n}\,dA = \alpha\cdot\dot{m}\,\frac{\bar{V}^2}{2}$

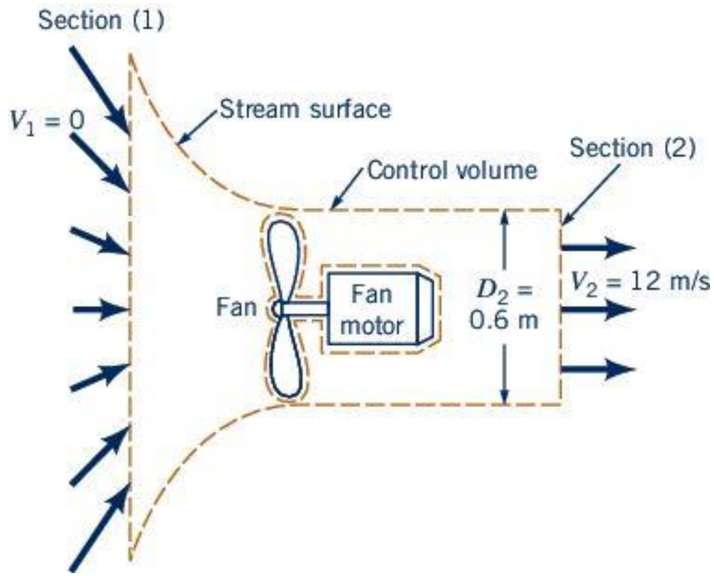
where $\bar{V}$ is the average velocity over the entry or exit surface.

α is dependent on the profile



Energy equation:

$$\frac{P_1}{\gamma} + \alpha_1\frac{\bar{V}_1^2}{2g} + z_1 + h_s = \frac{P_2}{\gamma} + \alpha_2\frac{\bar{V}_2^2}{2} + z_2 + h_L$$



The fan delivers 0.4 kW to the blades.
How much of the fan work produces useful effects, i.e. rise in energy?
What is the <u>fluid mechanical</u> <u>efficiency</u> of the fan?

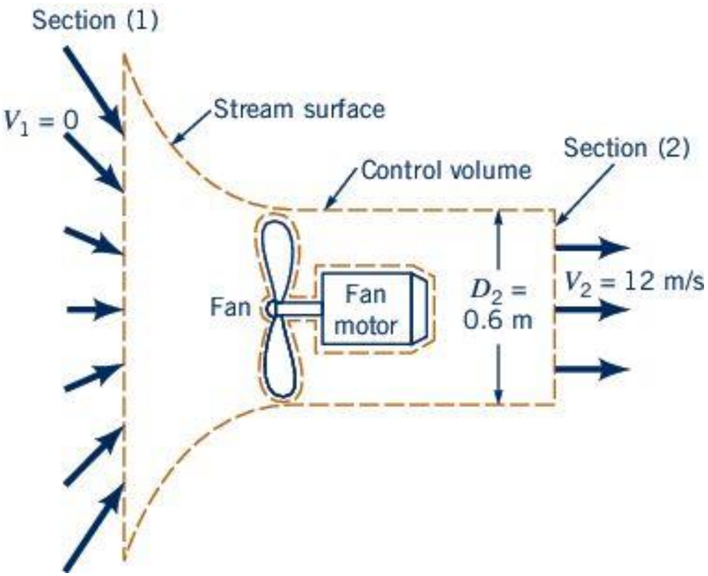
Energy between (1) and (2)

$$\underbrace{\cancel{\frac{P_1}{\gamma}}}_{0} + \underbrace{\cancel{\frac{V_1^2}{2g}}}_{\simeq 0} + \cancel{z_1} + h_p = \underbrace{\cancel{\frac{P_2}{\gamma}}}_{0} + \frac{V_2^2}{2g} + \cancel{z_2} + h_L \Rightarrow h_L = h_p - \frac{V_2^2}{2g} \qquad (1)$$

$$Q = V_2 \cdot A_2 = 12\ m/s \times \pi \times 0.3^2\ m^2 = \underline{3.4\ m^3/s}$$

$$\dot{W} = \gamma \cdot h_p \cdot Q \Rightarrow h_p = \frac{\dot{W}}{\gamma \cdot Q} = \frac{400\ W}{12\ N/m^3 \cdot 3.4\ m^3/s} \Rightarrow h_p = \underline{9.8\ m}$$

$$(1) \Rightarrow h_L = 9.8\ m - \frac{12^2\ m^2}{2 \times 9.81\ m/s^2} = \underline{2.46\ m}$$



The power lost in friction is

$$\dot{W}_L = h_L \cdot \gamma \cdot Q = 2.46\,m \cdot 12\,N/m^3 \times 3.4\,m^3/s$$
$$= 100.6\ W$$

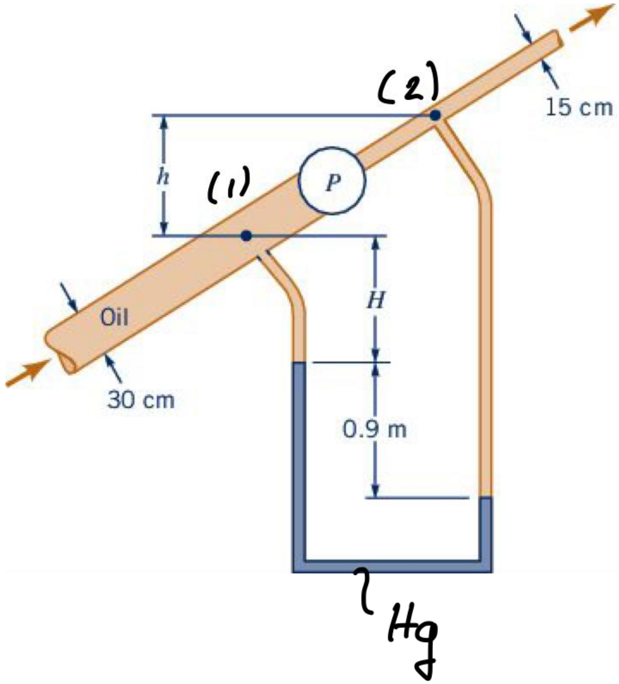
The useful power added to the system as kinetic energy is therefore

$$\dot{W}_{fluid} = \dot{W}_{fan} - \dot{W}_L = 400\ W - 100.6\ W$$
$$= \underline{299.4\ W}$$

The efficiency of the fan is

$$\eta = \frac{\dot{W}_{fluid}}{\dot{W}_{fan}} = \frac{299.4\ W}{400\ W} = 0.748$$

or $\underline{\underline{\eta \simeq 75\%}}$



$SG_{OIL} = 0.88 \qquad Q = 0.14\ m^3/s$

What is the power that the pump supplies to the oil?

Head losses are negligible

Energy between (1) and (2):

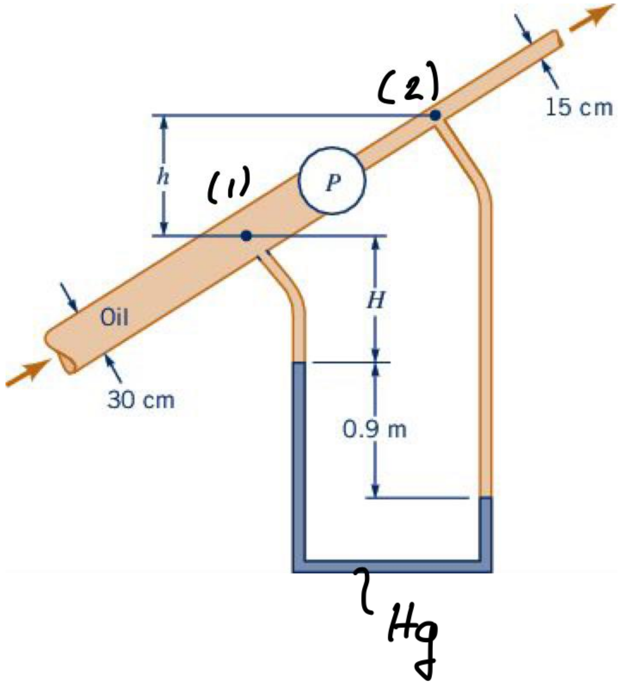
$$\frac{P_1}{\gamma_{OIL}} + \frac{V_1^2}{2g} + z_1 + h_p = \frac{P_2}{\gamma_{OIL}} + \frac{V_2^2}{2g} + z_2 + \cancel{h_L} \rightarrow 0$$

$$\Rightarrow h_p = \frac{P_2 - P_1}{\gamma_{OIL}} + \frac{V_2^2 - V_1^2}{2g} + \underbrace{z_2 - z_1}_{h} \qquad (1)$$

$$\gamma_{OIL} = SG_{OIL} \cdot \gamma_{H_2O} = 0.88 \times 9.8\ kN/m^3 = \underline{8.62\ kN/m^3}$$

$$V_1 = \frac{Q}{A_1} = \frac{0.14\ m^3/s}{\pi\ 0.15^2 m^2} = \underline{1.98\ m/s}$$

$$V_2 = \frac{Q}{A_2} = \frac{0.14\ m^3/s}{\pi \cdot 0.075^2 m^2} = \underline{7.92\ m/s}$$



Manometer equation:

$$P_1 + \gamma_{OIL}\cdot H + \gamma_{Hg}\cdot 0.9m - \gamma_{OIL}\,(0.9m + H + h) = P_2$$

$$\Rightarrow \quad P_2 - P_1 = (\gamma_{Hg} - \gamma_{OIL})\,0.9m - \gamma_{OIL}\cdot h$$

$$\Rightarrow \quad \frac{P_2 - P_1}{\gamma_{OIL}} = \left(\frac{\gamma_{Hg}}{\gamma_{OIL}} - 1\right) 0.9m - h \qquad (2)$$

Substituting (2) into (1):

$$h_p = \left(\frac{\gamma_{Hg}}{\gamma_{OIL}} - 1\right)\cdot 0.9m - \cancel{h} + \frac{V_2^2 - V_1^2}{2g} + \cancel{h}$$

$$= \left(\frac{\gamma_{Hg}}{\gamma_{OIL}} - 1\right) 0.9\,m + \frac{V_2^2 - V_1^2}{2g}$$

$$= \left(\frac{133.3\ kN/m^3}{8.62\ kN/m^3} - 1\right)\times 0.9\,m + \frac{7.92^2\ m^2/s^2 - 1.98^2\ m^2/s^2}{2\times 9.81\ m/s^2} = \underline{16\,m}$$

Finally the power of the pump is

$$\dot{W}_p = \gamma_{OIL}\cdot Q\cdot h_p = 8620\ N/m^3 \times 0.14\ m^3/s \times 16\,m = \underline{\underline{19'310\ W}}$$